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Consistency Of Definition Of Orthogonality In A Partial Metric Induced By A Metric

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Abstract:

An extension of the metric space in which the distance of the same point is not always zero is called a partial metric space. Orthogonality is the relation of two perpendicular lines at one point of intersection forming a right angle. There are several ways to define orthogonality, including Pythagorean Orthogonality, Isosceles Orthogonality, and Birkhoff-James Orthogonality. The purpose of this research is to study the consistency of the definition of orthogonality in the metric space to the partial metric space. Based on these results, it can be concluded that the partial metric space can be obtained by linear induction from a metric space. Then, in developing the definition of orthogonality to the partial metric space, it can be concluded that the qualified orthogonality is the I-orthogonality and the BJ-orthogonality, while the P-orthogonality does not qualify the consistency of the definition of orthogonality in the partial metric space.

Kata Kunci: *Orthogonality, Consistency, and Partial metric space*

1. INTRODUCTION

Maurice Frechet in 1906 first came up with the idea of a metric space (Frechet, 1906). Metric space is generally denoted by (X, d). The study about metric space continues in many ways. One of them is the generalization of metric spaces, namely partial metric spaces. The concep of the metric space is generalized in which the distance of the same point is not always zero. It is called a partial metric space. Matthews in 1992 first introduced the notion of partial metric spaces (Bukatin et al., 2009). Partial metric spaces were developed because of a problem in computer science, where two sequences that are equal and infinite do not necessarily have the same point that the distance is equal to zero. Therefore, it is easy to see the difference between metric and partial metric by looking at the distance between two similar points must be zero or not. In addition, it can be concluded that any metric is always a partial metric.

Matthews's research made many mathematicians conduct studies, proof, and develop the concept of partial metric spaces. Han, et al. (Han et al., 2017) in their research show the properties and principles of topology that apply in the metric partial space. The developed concept of the fixed-point theorem in metric partial space has been widely studied by researchers (see (Kir Mehmet, 2016)-(Bugajewski & Wang, 2020)). Study about compactness and completeness in metric partial space to research about orthogonality in metric space has also been investigated (see (Mykhaylyuk & Myronyk, 2019), (Senapati, 2018)).

Orthogonality is one of the interesting topics to be studied because orthogonality can be used as an easy alternative when looking for a vector space basis. In geometry, orthogonality is the relation of two perpendicular lines at one point of intersection and forming a right angle (Anton, 2011). In vector space, two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ on $\mathbb{R}^n$ are called orthogonal if $\boldsymbol{x} \cdot \boldsymbol{y} = 0$ (Gusnedi, 1999). In the norm space, there are several ways to define that two vectors are said to be orthogonal, including the Pythagorean

Orthogonality, Isosceles Orthogonality, and Birkhoff-James Orthogonality, each of them is denoted by $\boldsymbol{x} \perp_P \boldsymbol{y}$, $\boldsymbol{x} \perp_I \boldsymbol{y}$, and $\boldsymbol{x} \perp_{BJ} \boldsymbol{y}$ (Partington, 1986a). Orthogonality in the norm space is given different symbols to make it easier to check the definition of orthogonality, where $\boldsymbol{x} \perp_P \boldsymbol{y}$ represents the definition of Pythagorean Orthogonality, $\boldsymbol{x} \perp_I \boldsymbol{y}$ represents the definition of Isosceles Orthogonality, and $\boldsymbol{x} \perp_{BJ} \boldsymbol{y}$ represents the definition of Birkhoff-James Orthogonality. Research on orthogonality in vector space concludes that in vector space orthogonality, P-orthogonality, I-orthogonality, and BJ-orthogonality are all equivalent to the condition $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{0}$, which is usually symbolized by $\boldsymbol{x} \perp \boldsymbol{y}$.

A lot of research on orthogonality in spaces in mathematics have been carried out. There are studies about orthogonality that are used to assist the process of proving the existence of a fixed point in several orthogonal contractive type mappings (see: (Javed et al., 2021)-(Uddin et al., 2021)). Then there are several other researchers who develop a definition of orthogonality in a normed space to a certain space. In 2010, Jacek Chmielinski and Wojcik in their research estimated an orthogonality relation with the norm derivative (Chmieliński & Wójcik, 2010) In 2018, Jacek Chmielinski and Wojcik conducted another study in their research considering the approximate Birkhoff orthogonal symmetry and determining the orthogonality relation with some properties in X space (Chmieliński & Wójcik, 2018). Tawfeek, et al. In their research developed a J-orthogonality and Birkhoff orthogonality in smooth countably normed spaces (Tawfeek et al., 2021). Diminnie in his journal entitled "A New Orthogonality Relation for Normed Linear Spaces" studies the interesting properties of related property relation and uses these properties to find the characteristics of the inner product space (Diminnie, 1983).

Then Javed conducted research in his journal entitled "On Orthogonal Partial b-Metric Spaces with an Application" where in his research he studied the concept of orthogonality in partial b-metric spaces and ensured the existence of a single fixed point for several orthogonal contractive type mappings (Javed et al., 2021). The concept of orthogonality introduced by Javed is that two sets of X are said to be orthogonal in the partial metric space if they satisfy a binary relation which is notated by $(X, \perp)$, usually said to be an orthogonal set. However, the concept of orthogonality introduced by Javed only explains orthogonal sets, where the definition of orthogonality has not been specifically explained.

Inspired by Javed's research, the definition of orthogonality for partial metric is not an interesting metric to study. Since the definition of two vectors is said to be orthogonal in the partial metric space cannot be obtained directly, then one easy way to obtain a definition related to orthogonality in the partial metric space is to use an orthogonality definition that has been developed in the metric space. By using the definition of orthogonality in the metric space, a definition of orthogonality in the partial metric space will be formed and examined for the consistency of the orthogonality definition.

The term consistent appears in the context of solving linear equation system. A linear equation system is said to be consistent if the system has at least one solution (Anton, 2011). The term consistent can also be found in the context of a numerical partial differential equation system. Suppose given a partial differential equation system $P_u = f$ with a range of f is $\mathbb{R}$ and the numerical method for finding the solution of the system is $P_{\Delta_t,\Delta_x} v = f$. The numerical method for the differential equation system is said to be consistent if $|P\phi - P_{\Delta_t,\Delta_x}\phi| \to 0$ when $\Delta_t, \Delta_x \to 0$ for any smooth function $\phi(x,t)$ (William F. Ames, 2014). In the context of algebra, isomorphism preserves the structure or some properties of groups. Suppose if $f: G \to H$ is an isomorphic group then

G is an abelian group if and only if H is an abelian group (Joseph A Gallian, 2017). It can be said that the isomorphism consistently brings the characteristics of the G group, namely the abelian group, into the codomain of the isomorphism. So, the term consistency in this study refers to the term consistent in the context of algebra with some adjustments.

The focus in this study is to define two vectors that are orthogonal in a partial metric space that is not a metric space. Mentioning non-metric spaces is important because any metric space can be called a partial metric space. However, a partial metric space is not necessarily a metric space. As a result, the definition of orthogonality developed in the partial metric space must be consistent. What is meant by consistency is that two vectors that are orthogonal in the metric space must meet the orthogonal definition developed in the partial metric space.

After obtaining a consistent definition of orthogonality in the partial metric space, the next step is to study the properties of orthogonality in the partial metric space. The research conducted by Diminnie gave an idea to study the properties of orthogonality in the partial metric space and then use these properties to find characteristics related to orthogonality in the partial metric space. The orthogonality properties in the partial metric space that will be discussed in this study are set only for properties that satisfy and the continuity properties are not used, because the focus of this study does not discuss convergent sequences. The research in this article aims to study the consistency of the definition of orthogonality in the metric space to the partial metric space.

Based on the explanation above, the discussion section will explain about the partial metric space induced from a metric space. Next, we will discuss the definition of orthogonality in metric spaces. Then, using the definition of orthogonality in the metric space, a definition of orthogonality in the partial metric space will be formed and examined for the consistency of the orthogonality definition. Then it will also be examined for properties that satisfy orthogonality in the partial metric space.

2. THEORITICAL REVIEW

Bas Bas [10] said, there are several other ways to define that two vectors are said to be orthogonal in the norm space, including the Pythagorean Orthogonality, Isosceles Orthogonality, and Birkhoff-James Orthogonality, each of which is denoted by $\boldsymbol{x} \perp_{\boldsymbol{p}} \boldsymbol{y}, \boldsymbol{x} \perp_{\boldsymbol{I}} \boldsymbol{y}$, dan $\boldsymbol{x} \perp_{\boldsymbol{BJ}} \boldsymbol{y}$.

Definition 3 (Partington, 1986b). $\boldsymbol{x}$ said to be P-orthogonal to $\boldsymbol{y}$ denoted by ($\boldsymbol{x} \perp_P \boldsymbol{y}$) if and only if $\|\boldsymbol{x}-y\|^2 = \|\boldsymbol{x}\|^2 + \|y\|^2$.

Definition 4 (Partington, 1986a). $\boldsymbol{x}$ said to be I-orthogonal to $\boldsymbol{y}$ denoted by ($\boldsymbol{x} \perp_I \boldsymbol{y}$) if and only if $\|\boldsymbol{x}+y\| = \|\boldsymbol{x}-\boldsymbol{y}\|$.

Definition 5 (Partington, 1986a). $\boldsymbol{x}$ said to be BJ-orthogonal to $\boldsymbol{y}$ denoted by ($\boldsymbol{x} \perp_{BJ} \boldsymbol{y}$) if and only if $\|\boldsymbol{x}+\alpha\boldsymbol{y}\| \geq \|\boldsymbol{x}\|$ for each $\alpha \in \mathbb{R}$.

Definition 6 (Matthews. S.G, 1994). Let X be a non empty set. A function $p: X \times X \to \mathbb{R}$ is called a partial metric in X if for every $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$, it holds:

(P1) $p(\boldsymbol{x},\boldsymbol{x}) \leq p(\boldsymbol{x},\boldsymbol{y})$.

(P2) if $p(\boldsymbol{x},\boldsymbol{x}) = p(\boldsymbol{x},\boldsymbol{y}) = p(\boldsymbol{y},\boldsymbol{y})$ then $\boldsymbol{x} = \boldsymbol{y}$.

(P3) $p(\boldsymbol{x},\boldsymbol{y}) = p(\boldsymbol{y},\boldsymbol{x})$.

(P4) $p(\boldsymbol{x},\boldsymbol{y}) \leq p(\boldsymbol{x},\boldsymbol{z}) + p(\boldsymbol{z},\boldsymbol{y}) - p(\boldsymbol{z},\boldsymbol{z})$.

So p is said to be a partial metric in X and (X,p) is a partial metric space.

3. METHOD

This research method is a deductive. Several examples are taken into account. From several metric space, it is proven or disprove whether it is partial metric space or not and vice versa. Then it is generalized.

4. RESULTS AND DISCUSSION

In the results and discussion section, we will review several definitions and theorems related to partial metric space, orthogonality of partial metric space, and properties that satisfy the orthogonality of partial metric space. In addition, this section will describe the consistency theorem on the definition of orthogonality in the metric space to the partial metric space. Before discussing consistency of definition of orthogonality in the metric space to the partial metric space, this section will introduce the partial metric space theorem induced from the metric space.

Theorems 2: partial metric space induced from metric space.

Given a metric space (X,d). If $p(\boldsymbol{x},\boldsymbol{y}) = c_1 d(\boldsymbol{x},\boldsymbol{y}) + c_2$ which $\boldsymbol{x},\boldsymbol{y} \in X$ and c_1, c_2 is a real constant with $c_1 > 0$, then (X,p) is partial metric space.

Proof

We will show that $p(\boldsymbol{x},\boldsymbol{x}) \leq p(\boldsymbol{x},\boldsymbol{y})$ for every $\boldsymbol{x},\boldsymbol{y} \in X$. Choose any $\boldsymbol{x},\boldsymbol{y} \in X$, then $p(\boldsymbol{x},\boldsymbol{x}) = c_1 d(\boldsymbol{x},\boldsymbol{x}) + c_2$. We know that in metric space $d(\boldsymbol{x},\boldsymbol{x}) = 0$, and $d(\boldsymbol{x},\boldsymbol{y}) \geq 0$ then $p(\boldsymbol{x},\boldsymbol{x}) = c_2 \leq c_1 d(\boldsymbol{x},\boldsymbol{y}) + c_2 = p(\boldsymbol{x},\boldsymbol{y})$, where c_1 nonnegative. So $p(\boldsymbol{x},\boldsymbol{x}) \leq p(\boldsymbol{x},\boldsymbol{y})$ for every $\boldsymbol{x},\boldsymbol{y} \in X$.

We will show that if $p(\boldsymbol{x},\boldsymbol{x}) = p(\boldsymbol{x},\boldsymbol{y}) = p(\boldsymbol{y},\boldsymbol{y})$, then $\boldsymbol{x} = \boldsymbol{y}$. Choose any $\boldsymbol{x},\boldsymbol{y} \in X$, then

$p(\boldsymbol{x},\boldsymbol{x}) = p(\boldsymbol{x},\boldsymbol{y}) = p(\boldsymbol{y},\boldsymbol{y})$

$\Leftrightarrow c_1 d(\boldsymbol{x},\boldsymbol{x}) + c_2 = p(\boldsymbol{x},\boldsymbol{y}) = c_1 d(\boldsymbol{y},\boldsymbol{y}) + c_2$

$\Leftrightarrow c_2 = p(\boldsymbol{x},\boldsymbol{y}) = c_2$

$\Leftrightarrow c_2 = c_1 d(\boldsymbol{x},\boldsymbol{y}) + c_2 = c_2$

$\Leftrightarrow 0 = c_1 d(\boldsymbol{x},\boldsymbol{y}) = 0.$ **(1)**

After that we divide all three sides in equation (1) with a constat $c_1 > 0$, then

$$0 - d(\boldsymbol{x},\boldsymbol{y}) - 0.$$

We know in metric space that $d(\boldsymbol{x},\boldsymbol{y}) = 0$ if and only if $\boldsymbol{x} = \boldsymbol{y}$. Then we get $\boldsymbol{x} = \boldsymbol{y}$. So, if $p(\boldsymbol{x},\boldsymbol{x}) = p(\boldsymbol{x},\boldsymbol{y}) = p(\boldsymbol{y},\boldsymbol{y})$ then $\boldsymbol{x} = \boldsymbol{y}$ for every $\boldsymbol{x},\boldsymbol{y} \in X$.

We will show that $p(\boldsymbol{x},\boldsymbol{y}) = p(\boldsymbol{y},\boldsymbol{x})$ for every $\boldsymbol{x},\boldsymbol{y} \in X$. Choose any $\boldsymbol{x},\boldsymbol{y} \in X$, then

$$p(\boldsymbol{x},\boldsymbol{y}) = c_1 d(\boldsymbol{x},\boldsymbol{y}) + c_2 = c_1 d(\boldsymbol{y},\boldsymbol{x}) + c_2 = p(\boldsymbol{y},\boldsymbol{x}).$$

So $p(\boldsymbol{x},\boldsymbol{y}) = p(\boldsymbol{y},\boldsymbol{x})$ for every $\boldsymbol{x},\boldsymbol{y} \in X$.

We will show that $p(\boldsymbol{x},\boldsymbol{y}) \leq p(\boldsymbol{x},\boldsymbol{z}) + p(\boldsymbol{z},\boldsymbol{y}) - p(\boldsymbol{z},\boldsymbol{z}), \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in X$. Choose $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in X$, given d is a metric in X. So that there is satisfy triangular inequality

$$d(\boldsymbol{x},\boldsymbol{y}) \leq d(\boldsymbol{x},\boldsymbol{z}) + d(\boldsymbol{z},\boldsymbol{y}), \qquad \textbf{(2)}$$

We know that in metric space

$$d(\boldsymbol{z},\boldsymbol{z}) = 0. \qquad \textbf{(3)}$$

Then

$$d(\boldsymbol{x},\boldsymbol{y}) \leq d(\boldsymbol{x},\boldsymbol{z}) + d(\boldsymbol{z},\boldsymbol{y}) - 0. \qquad \textbf{(4)}$$

After that, we subtitute equation (3) to equation (4) then

$$d(\boldsymbol{x},\boldsymbol{y}) \leq d(\boldsymbol{x},\boldsymbol{z}) + d(\boldsymbol{z},\boldsymbol{y}) - d(\boldsymbol{z},\boldsymbol{z}). \qquad \textbf{(5)}$$

Then we multiply both side in equation (5) with constant c_1, and we get

$$c_1 d(\boldsymbol{x},\boldsymbol{y}) \leq c_1\big(d(\boldsymbol{x},\boldsymbol{z}) + d(\boldsymbol{z},\boldsymbol{y}) - d(\boldsymbol{z},\boldsymbol{z})\big)$$

$$\Leftrightarrow c_1 d(\boldsymbol{x},\boldsymbol{y}) \leq c_1 d(\boldsymbol{x},\boldsymbol{z}) + c_1 d(\boldsymbol{z},\boldsymbol{y}) - c_1 d(\boldsymbol{z},\boldsymbol{z}). \qquad \textbf{(6)}$$

Then we add constat c_1 to the both side in equation (5) so

$$c_1 d(x,y) + c_2 < c_1 d(x,z) + c_2 + c_1 d(z,y) + c_2 - (c_1 d(z,z) + c_2). \qquad \textbf{(7)}$$

Because $c_1 d(\boldsymbol{x},\boldsymbol{y}) + c_2 = p(\boldsymbol{x},\boldsymbol{y})$ the we can write equation (7) as

$$p(\boldsymbol{x},\boldsymbol{y}) \leq p(\boldsymbol{x},\boldsymbol{z}) + p(\boldsymbol{z},\boldsymbol{y}) - p(\boldsymbol{z},\boldsymbol{z}).$$

So we know that $p(\boldsymbol{x},\boldsymbol{y}) \leq p(\boldsymbol{x},\boldsymbol{z}) + p(\boldsymbol{z},\boldsymbol{y}) - p(\boldsymbol{z},\boldsymbol{z})$ for every $\boldsymbol{x},\boldsymbol{y} \in X$.

Its proved that (X,p) is a partial metric space.

Definition 7. Given a metric space (X,d). If $p(\boldsymbol{x},\boldsymbol{y}) = c_1 d(\boldsymbol{x},\boldsymbol{y}) + c_2$ which $\boldsymbol{x},\boldsymbol{y} \in X$ and c_1, c_2 is a real constant with $c_1 > 0$, then p is a partial metric space that linearly inducted from metric d.

Next, we will examine the definition of orthogonality in metric spaces. One easy way to obtain a definition of orthogonality in the metric space is using definition of orthogonality in the norm space. The definition of orthogonality in the norm space is developed to the metric space by using the relation $d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$. Thus, the following definitions are obtained.

Definition 8. Let (X,d) is a metric space, for every $\boldsymbol{x},\boldsymbol{y} \subset X$, $\boldsymbol{x}$ said to be ortoghonal-*P* to $\boldsymbol{y}$ and denoted by $(\boldsymbol{x} \perp_{P_1} \boldsymbol{y})$ if and only if

$$\big(d(\boldsymbol{x},\boldsymbol{y})\big)^2 - \big(d(\boldsymbol{x},\boldsymbol{0})\big)^2 + \big(d(\boldsymbol{y},\boldsymbol{0})\big)^2.$$

Definition 9. Let (X,d) is a metric space, for every $\boldsymbol{x},\boldsymbol{y} \in X$, $\boldsymbol{x}$ said to be I-orthogonal to $\boldsymbol{y}$ and denoted by $(\boldsymbol{x} \perp_{I_1} \boldsymbol{y})$ if and only if

$$d(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{0}) = d(\boldsymbol{x},\boldsymbol{y}).$$

Definition 10. Let (X,d) is a metric space, for every $\boldsymbol{x},\boldsymbol{y} \in X$, $\boldsymbol{x}$ said to be BJ-orthogonal to $\boldsymbol{y}$ and denoted by $(\boldsymbol{x} \perp_{BJ_1} \boldsymbol{y})$ if and only if

$d(\boldsymbol{x}+\alpha \boldsymbol{y}, 0) \geq d(\boldsymbol{x}, \mathbf{0})$.

After obtaining the definition of orthogonality in the metric space, then a definition of orthogonality in the partial metric space is formed and its consistency is examined. However, in the definition of P-orthogonality in the metric space when it is developed to a partial metric space, a definition is obtained, that is, Let (X,p) is a partial metric space, for every $\boldsymbol{x}, \boldsymbol{y} \in X$, $\boldsymbol{x}$ said to be P-orthogonal to $\boldsymbol{y}$ and denoted by $(\boldsymbol{x} \perp_{P_2} \boldsymbol{y})$ if and only if $\left(p(\boldsymbol{x},\boldsymbol{y})\right)^2 = \left(p(\boldsymbol{x},\mathbf{0})\right)^2 + \left(p(\boldsymbol{y},\mathbf{0})\right)^2$. However, the definition is inconsistent as indicated by the following note 2. Furthermore, it will be discussed about the development of the definition of I-orthogonal in the partial metric space, its consistency, and some of its properties.

Definition 11. Let (X,p) is a metric space, for every $\boldsymbol{x}, \boldsymbol{y} \in X$, $\boldsymbol{x}$ said to be I-orthogonal to $\boldsymbol{y}$ and denoted by $(\boldsymbol{x} \perp_{I_2} \boldsymbol{y})$ if and only if

$$p(\boldsymbol{x}+\boldsymbol{y}, \mathbf{0}) = p(\boldsymbol{x}, \boldsymbol{y}).$$

Theorem 3 : I-Orthogonality Definition Consistency in Metric Space to Partial Metric Space.

If $\boldsymbol{x} \perp_{I_1} \boldsymbol{y}$ in a metric space (X,d), then $\boldsymbol{x} \perp_{I_2} \boldsymbol{y}$ in a partial metric space (X,p) with p is partial metric that linearly inducted from d metric

Proof

Given $\boldsymbol{x} \perp_{I_1} \boldsymbol{y}$ in a metric space, so that

$$d(\boldsymbol{x}+\boldsymbol{y}, \mathbf{0}) = d(\boldsymbol{x}, \boldsymbol{y}). \quad \textbf{(8)}$$

It will be shown that $\boldsymbol{x} \perp_{I_2} \boldsymbol{y}$ in partial metric space that linearly inducted from d metric.

Note that,

$$p(\boldsymbol{x}+\boldsymbol{y}, \mathbf{0}) - c_1 d(\boldsymbol{x}+\boldsymbol{y}, \mathbf{0}) + c_2$$

$$= c_1 d(\boldsymbol{x}, \boldsymbol{y}) + c_2$$

$$= p(\boldsymbol{x}, \boldsymbol{y})$$

For $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$.

Then it is proved that $p(\boldsymbol{x}+\boldsymbol{y}, \mathbf{0}) = p(\boldsymbol{x}, \boldsymbol{y})$. So, if $\boldsymbol{x} \perp_{I_1} \boldsymbol{y}$ in a metric space (X,d), then $\boldsymbol{x} \perp_{I_2} \boldsymbol{y}$ in a partial metric space (X,p) with p is partial metric that linearly inducted from d metric.

Theorem 4 : I-Orthogonality Properties in Partial Metric Spaces.

Let (X,p) is a partial metric space with p linearly inducted from a metric, then $\perp_{I_2}$ in (X,p) satisfy the non-degenerate and symmetrical properties.

Proof It will be shown that $\perp_{I_2}$ in (X,p) satisfy the non-degenerate and symmetrical properties.

a. Non-degenerate properties.

Because $\mathbf{x} \perp_{I_2} \mathbf{x}$, then applies

$p(\mathbf{x}+\mathbf{x},\mathbf{0}) = p(\mathbf{x},\mathbf{x})$

$\Leftrightarrow c_1 d(\mathbf{x}+\mathbf{x},\mathbf{0}) + c_2 = c_1 d(\mathbf{x},\mathbf{x}) + c_1$

$\Leftrightarrow d(2\mathbf{x},\mathbf{0}) = 0$

$\Leftrightarrow 2\mathbf{x} = \mathbf{0}$

$\leftrightarrow \mathbf{x} - \mathbf{0}$

for $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$.

So $\perp_{I_2}$ in partial metric space satisfy non-degenerative properties.

b. Symetrical properties.

We have $\mathbf{x} \perp_{I_2} \mathbf{y}$, then

$p(\mathbf{x}+\mathbf{y},\mathbf{0}) = p(\mathbf{x},\mathbf{y})$

$\Leftrightarrow c_1 d(\mathbf{x}+\mathbf{y},\mathbf{0}) + c_2 = c_1 d(\mathbf{x},\mathbf{y}) + c_2$

$\Leftrightarrow d(\mathbf{x}+\mathbf{y},\mathbf{0}) = d(\mathbf{x},\mathbf{y})$

$\Leftrightarrow d(\mathbf{y}+\mathbf{x},\mathbf{0}) = d(\mathbf{y},\mathbf{x})$

$\Leftrightarrow c_1 d(\mathbf{y}+\mathbf{x},\mathbf{0}) + c_2 = c_1 d(\mathbf{y},\mathbf{x}) + c_2$

$\Leftrightarrow p(\mathbf{y}+\mathbf{x},\mathbf{0}) = p(\mathbf{y},\mathbf{x})$

for $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$.

Then obtained $\mathbf{y} \perp_{I_2} \mathbf{x}$.

So $\perp_{I_2}$ in partial metric space is satisfy symmetrical properties. Furthermore, it will be discussed about the development of the definition of BJ-orthogonal in the partial metric space, its consistency, and some of its properties.

Definition 12.

Let (X,p) is a partial metric space, for every $\mathbf{x},\mathbf{y} \in X$, $\mathbf{x}$ said to be BJ-orthogonal to $\mathbf{y}$ and denoted by $(\mathbf{x} \perp_{BJ_2} \mathbf{y})$ if and only if

$p(\boldsymbol{x}+\alpha\boldsymbol{y},\boldsymbol{0}) \geq p(\boldsymbol{x},\boldsymbol{0}), \quad \forall\alpha \in \mathbb{R}.$

Theorem 5 : Consistency of BJ-Orthogonality Definition in Metric Space to Partial Metric Space.

If $\boldsymbol{x} \perp_{BJ_1} \boldsymbol{y}$ in a metric space (X,d), then $\boldsymbol{x} \perp_{BJ_2} \boldsymbol{y}$ in apartial metric space (X,p) with p is a partial metrik that linearly inducted from d metric.

Proof

Given $\boldsymbol{x} \perp_{BJ_1} \boldsymbol{y}$ in metric space, then applies

$d(\boldsymbol{x}+\alpha\boldsymbol{y},\boldsymbol{0}) \geq d(\boldsymbol{x},\boldsymbol{0}).$ **(9)**

It will be shown that $\boldsymbol{x} \perp_{BJ_2} \boldsymbol{y}$ in partial metric space that linearly inducted from d metric.

Note that

$p(\boldsymbol{x}+\alpha\boldsymbol{y},\boldsymbol{0}) = c_1 d(\boldsymbol{x}+\alpha\boldsymbol{y},\boldsymbol{0}) + c_2$

$\geq c_1 d(\boldsymbol{x},\boldsymbol{0}) + c_2$

$= p(\boldsymbol{x},\boldsymbol{0})$

for $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$.

Then it is proven that $p(\boldsymbol{x}+\alpha\boldsymbol{y},\boldsymbol{0}) > p(\boldsymbol{x},\boldsymbol{0})$.

So, if $\boldsymbol{x} \perp_{BJ_1} \boldsymbol{y}$ in a metric space (X,d), then $\boldsymbol{x} \perp_{BJ_2} \boldsymbol{y}$ in a partial metric space (X,p) which p is a partial metric that linearly induced by d metric.

Theorem 6 : Properties of BJ-Orthogonality in Partial Metric Spaces.

Let (X,p) is a partial metric space, then $\perp_{BJ_2}$ in a partial metric space is datisfy non-degenerative properties.

Proof

It will be shown that $\perp_{BJ_2}$ in partial metric space satisfy non-degenerative propeperties.

that is:

if $\boldsymbol{x} \perp_{BJ_2} \boldsymbol{x}$, then $\boldsymbol{x} = \boldsymbol{0}$.

Given $\boldsymbol{x} \perp_{BJ_2} \boldsymbol{x}$, then

$p(\boldsymbol{x}+\alpha\boldsymbol{x},\boldsymbol{0}) \geq p(\boldsymbol{x},\boldsymbol{0})$

$\Leftrightarrow c_1 d(\boldsymbol{x}+\alpha\boldsymbol{x},\boldsymbol{0}) + c_2 \geq c_1 d(\boldsymbol{x},\boldsymbol{0}) + c_2$

$\Leftrightarrow d(\boldsymbol{x}+\alpha\boldsymbol{x},\mathbf{0}) \geq d(\boldsymbol{x},\mathbf{0})$, for every $\alpha \in \mathbb{R}$, and for $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$.

Because it applies for every $\alpha \in \mathbb{R}$, we can choose $\alpha = -\frac{1}{2}$. Then the result is,

$$d\left(\frac{1}{2}\boldsymbol{x},\mathbf{0}\right) \geq d(\boldsymbol{x},\mathbf{0}). \quad \textbf{(10)}$$

Inequality (10) applies only for $\boldsymbol{x} = \mathbf{0}$.

So $\perp_{BJ_s}$ in partial metric space satisfies non-degenerative properties.

5. CONCLUSION

In this research, it can be concluded that a partial metric space can be induced linearly from a metric. Then, in developing the definition of orthogonality to the partial metric space, it can be concluded that the orthogonality that satisfies the I-orthogonality and the BJ-orthogonality, while the P-orthogonality does not satisfy the consistency of the definition of orthogonality in the partial metric space. Based on the results of this research, the author suggests to readers or further researchers to be able to develop research related to orthogonality in generalizations of other partial metric spaces, because there are still many other properties that need to be investigated and assessed for their applicability.

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