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Max Plus Algebra Application In Air Defence Systems

Zumrotus Sya'diyah
Program Studi Pendidikan Matematika, STKIP Gotong Royong Masohi
email: zuma.yakuza@gmail.com

Abstract:

Air defence is one of the vital systems in national defence. This system generally includes several processes, namely detection by radar, object identification and decision making. The level of success really depends on the accuracy of decision making. However, it is not yet known for certain which processes critical processes in decision are making. In this research, we will discuss the application of max plus algebra in determining which process of the workflow for air defence systems is the most critical one. Graphical interpretation of the system using graphs and Timed Petri Nets to make it easier to understand and analyse. The research results show that the success of the entire process depends on the readiness of the radar in carrying out surveillance of the coverage area. This can be seen from the critical circuit which only depends on the place relating to the situation.

Keywords**:** *Max plus algebra, Timed Petri Net, air defence system.*

1. INTRODUCTION

The air security system is an important component in maintaining the sustainability of the country. This security system consists of early detection processes, identification of flying objects, movement tracking. Decision making for rapid intervention (via radar, satellite, communications system or short notice) (Yang, 2022).

Discussions related to military systems are mostly analysed using another than algebra method. Such as a research in the term of missile defence usage or destruction (Wilkening, 2000). This research determine the number of ballistic missile defense interceptors with a specific defense objective using probabilistic method. Another research is held in this fields is the air defence plan intersection using A* optimation algorithm (Song et al., 2023). In this research the interception-cost mixed optimal function of the system optimization objective is obtained. There is also studies about the air defence operation system. Both of them are using swarm method in its analysing process, one is held based on Australian military operation (Mcilroy et al., 2022) and the other is using US Air Force operational system .

The study of air defence system especially in terms of mathematical algebra is still minimally discussed. While more points of view on this topic will provide more options for optimization in its operational system. So more varied mathematical models are needed from all points of view in order to enrich references and add alternatives for choosing the most appropriate model. In this research, we use the max plus algebra theory combined with timed petri net to analyse the operation of air defence system, which has not been studied before (Jackson, Jack A, Jones Brian L, 2025).

As we know that the level of success of the air security system is very dependent on the speed and accuracy of decision making, especially in scheduling and coordination of all defense system units involved (Zhao, 2024). However, this very crucial process in decision making cannot yet be determined with certainty. So far, the assumption used in the operation of air defense systems is that the entire process is a crucial process in its influence on decision making.

Therefore, a proper analysing is needed to determine the crucial processes that have the most influence on decision making. These problem can be clearly given by analysing this system using max plus algebra combined with timed petri net. This is because max plus algebra has been proven to be very effective in analysing the procedural problem, such as scheduling and manufacturing, modeling but has not been widely applied in military and security systems. This research will model the air security system workflow into max plus algebraic equations to identify critical circuits (critical processes) that contribute directly to decision making.

2. METHODS

In this research, the flow (how it works) of the air defense system will be expressed in a Timed Petri Net graph which will then be modeled in the form of a max plus algebraic matrix equation. Next, the critical process is determined based on the results of determining the eigenvalues. This model was created without using real time. This is intended so that the model can more accurately determine critical processes in general (paying attention to the process flow for the system in general). The eigenvalue determination uses the average weight of the maximum circuit (Sya'diyah, 2023) and is only a simulation considering that the real time for each process will be different and classified for each type of system (U.S. Air Force, 2021).

The research steps are as follows:

1. Conduct a literature review related to Max-Plus algebra and air defense systems, including the previous research.
2. Develop a flowchart of the air defense system.
3. Built the Petri Net of the air defense system.
4. Formulate the Max-Plus algebra model of the air defense system.
5. Construct the Max-Plus algebra matrix based on the developed model.
6. Solve the eigenvalue problem of the Max-Plus matrix.
7. Identify the critical process of the constructed system.

Those steps can be seen more accurately in the Figure 1.

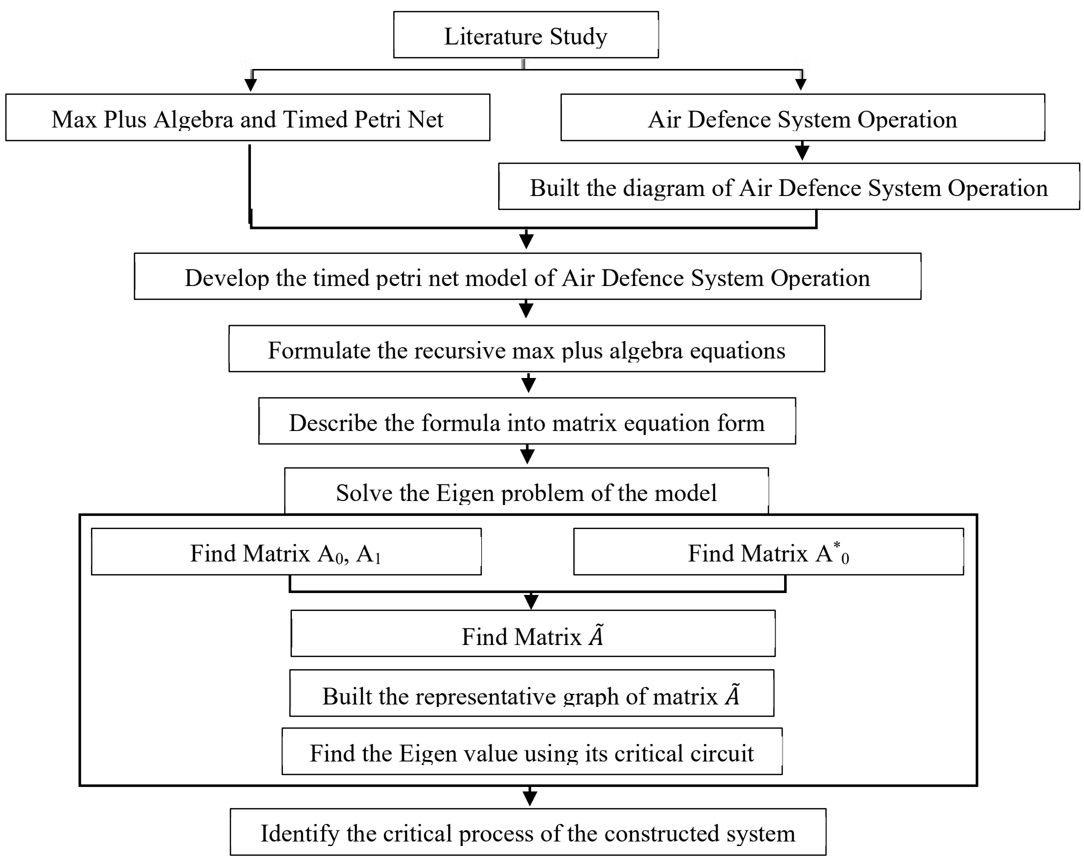

Figure 1. Diagram of the Research Method

Figure 1 shows us that the eigen problem can be solved through 4 steps:

1. Determine the matrix A_0 and A_1 from the max plus algebra equations.
2. Calculate the matrix $\widetilde{A}$ using the theory of autonomous equations of max plus algebra.
3. Develop a graph according to matrix $\widetilde{A}$ obtained.
4. Find the critical circuit to determine the eigen value of the system. This step have to be done without using any algorithm of sigenmode finding because all of the holding times in this research is not determined explicitely.

Furthermore, the following theory is the basic theory and methods we used in modeling the air security system in this research.

2.1. Max Plus Algebra

Scheduling is the process of determining the optimal time for carrying out activities. In the context of discrete dynamical systems, Max-Plus algebra is a mathematical framework used to model systems in which process duration and time dependencies are important elements, such as in production systems, transportation, or computer networks.

The domain used in the discussion of max plus algebra is expanded real numbers with $\varepsilon = -\infty$. Next is this set of numbers, $\mathbb{R} \cup \{\varepsilon\}$, notated by $\mathbb{R}_\varepsilon$. Where this set is a semiring with neutral elements ε and its unit element is 0. In set of $\mathbb{R}_\varepsilon$ defined several operations. For each $x, y \in \mathbb{R}_\varepsilon$

$$x \oplus y \stackrel{\text{def}}{=} \max\{x, y\}$$

$$x \otimes y \stackrel{\text{def}}{=} x + y$$

From operation $\otimes$ then the following operation can be defined:

$$x^{\otimes \alpha} = \alpha\, x \text{ , dengan } \alpha \in \mathbb{R}_\varepsilon$$

The set of $\mathbb{R}_\varepsilon$ together with the operations $\oplus$ and $\otimes$, i.e. $(\mathbb{R}_\varepsilon, \oplus, \otimes)$, notated by $\mathbb{R}_{max}$. A more reliable explanation of max plus algebra can be seen in (Sya'diyah, 2023), (Al Bermanei et al., 2024), (Ooga et al., 2024), (Zhang & Zhu, 2025), and (Nishida, 2024).

2.2. Petri Net

Petri Net is a 4-tuple (P, T, A, w) where P is a finite set of *place*, i.e. $P = \{P_1, P_2, \cdots, P_n\}$, T is a finite set of transition, i.e. $T = \{t_1, t_2, \cdots, t_n\}$, and A is a set of *arc*, i.e. $A \subseteq (P \times T) \cup (T \times P)$. Whereas w is weight function, i.e. $w: A \to \{1,2,3, \cdots\}$ (Sya'diyah, 2023).

Petri Net graphs consist of two kinds of shapes, namely lines (rectangles) and circles. Lines (rectangles) represent transitions, circles represent places, while arcs are expressed by arrows connecting places with transitions. For example, the weight of the arc leading from *place*, p_i to the transition of t_j written by $w(p_i, t_j) = k$. This means that there are k of *arcs* from *place* p_i to the transition of t_j (Bahri et al., 2003).

In the terms of Petri net, the following notation is also used:

$$I(t_j) = \{p_i : (p_i, t_j)\},$$

$$O(t_j) = \{p_i : (t_j, p_i)\}$$

Furthermore, we will give a definition of a pure Petri Net. A Petri Net is said to be pure if the Petri Net has places that are both input and output for a transition. Meanwhile, the Timed Petri Net in this research is a development of the Petri Net definition. Timed

Petri Net is a Petri Net where the places or transitions have holding times. In this research, holding times are used for all places in the Petri Net. A more in-depth theory can be obtained at (Jarabo et al., 2024), (Radom & Formanowicz, 2024), (Strzęciwilk, 2023), dan (Shailesh et al., 2020).

3. RESULT AND DISCUSSION

According to (Zhao, 2024), (Dantas et al., 2023), (Yang, 2022), and (U.S. Air Force, 2021) the flow or working method of the air defence system in general is as follows:

1. Radar detects the entire surveillance area
2. If there is a flying object, the radar detects it and categorizes it into 3 types:
   i. Unknown objects are then considered a threat
   ii. The object is recognized and considered a serious threat
   iii. The object is recognized and deemed not to be a threat
3. If the object is considered a threat, the control system will make a decision to launch a missile/missile to destroy the object
4. If the object is considered a mild or moderate threat, the control system will make a decision to neutralize the object

The operation process of air defence system above can be describe into a diagram as we seen in **Figure 2** below.

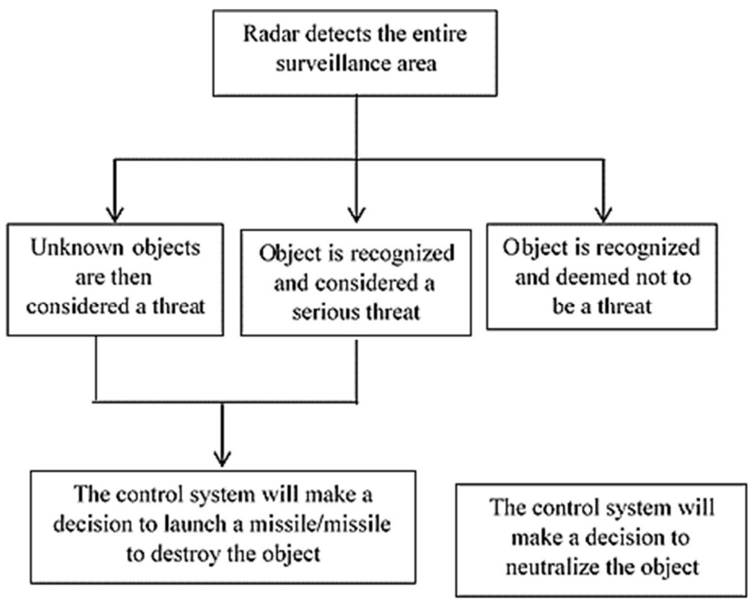

Figure 2. Diagram of Air Defence System

From the steps and processes above, a Timed Petri Net graph can be constructed as shown in **Figure 3**. From this image, it can be seen that the Timed Petri Net graph formed consists of 9 places and 7 transitions. Where each place and transition denotes a state and occurrence as follows:

P_0 = Radar is idle
P_1 = Target enters surveillance area
P_2 = Target detected by radar
P_3 = Target is defined as unknown object or target with threat
P_4 = Target has been decided to be destroyed
P_5 = Target has been destroyed

P_6 = Target is defined as an object without a threat
P_7 = Target has been decided to be neutralized
P_8 = Target has been neutralized
t_0 = Radar detected the surveillance area
t_1 = Radar detects target
t_2 = The system identifies the target
t_3 = Decision making by the system (target destruction)
t_4 = Delivery of missiles/destructive missiles
t_5 = Decision making by the system (target neutralization)
t_5 = Target neutralization

From **Figure 3** the max plus algebraic recursive equation can be formed as the following equations:

$$x_0(k) = C_0 \otimes x_0(k-1)$$

$$x_1(k) = C_1 \otimes x_0(k)$$

$$x_2(k) = C_2 \otimes x_1(k)$$

$$x_3(k) = C_3 \otimes x_2(k)$$

$$x_4(k) = C_4 \otimes x_3(k)$$

$$x_5(k) = C_6 \otimes x_2(k)$$

$$x_6(k) = C_7 \otimes x_5(k)$$

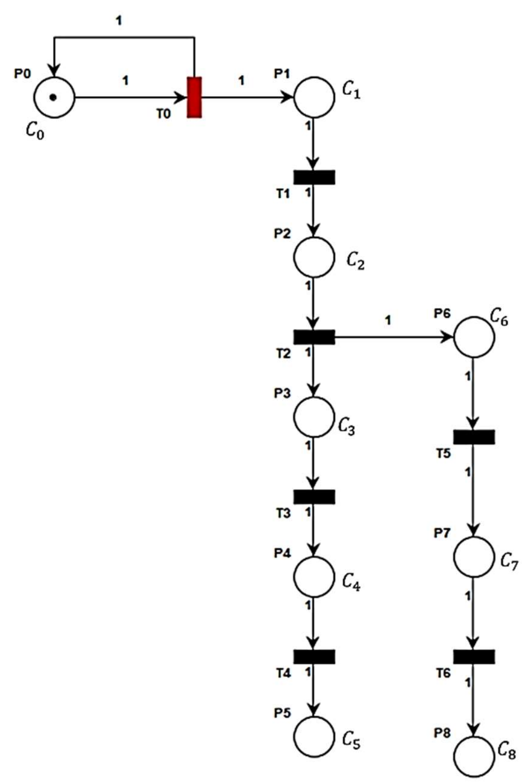

Figure 3. *Timed Petri Net from Air Security System Flow*

From the recursive equation above, the following matrix is obtained:

$$A_0 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ C_1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & C_2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & C_3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & C_4 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & C_6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & C_7 & \varepsilon \end{bmatrix} \text{ dan } A_1 = \begin{bmatrix} C_0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}$$

Furthermore, the max plus algebraic autonomous system matrix is calculated as follows (Sya'diyah, 2023), (Carnia et al., 2023) and (Zhang & Zhu, 2025):

$$\tilde{A} = A_0^{*} \otimes A_1$$

Where it can be determined that:

$$A_0^{*} = \oplus_{i=0}^{n-1} (A_0)^{\otimes i} = (A_0)^{\otimes 1} \oplus (A_0)^{\otimes 2} \cdots \oplus (A_0)^{\otimes 6}$$

$$A_0^{*} = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ C_1 + C_2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ C_1 + C_2 + C_3 & C_2 + C_3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ C_1 + C_2 + C_3 + C_4 & C_2 + C_3 + C_4 & C_3 + C_4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ C_1 + C_2 + C_6 & C_2 + C_6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ C_1 + C_2 + C_6 + C_7 & C_2 + C_6 + C_7 & C_6 + C_7 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}$$

Therefore, we can be obtain that:

$$\tilde{A} = A_0^{*} \otimes A_1 = \begin{bmatrix} C_0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ C_1 + C_2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ C_1 + C_2 + C_3 & C_2 + C_3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ C_1 + C_2 + C_3 + C_4 & C_2 + C_3 + C_4 & C_3 + C_4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ C_1 + C_2 + C_6 & C_2 + C_6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ C_1 + C_2 + C_6 + C_7 & C_2 + C_6 + C_7 & C_6 + C_7 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}$$

From the matrix of $\tilde{A}$ above, the corresponding directed graph can be formed as shown in **Figure 4**.

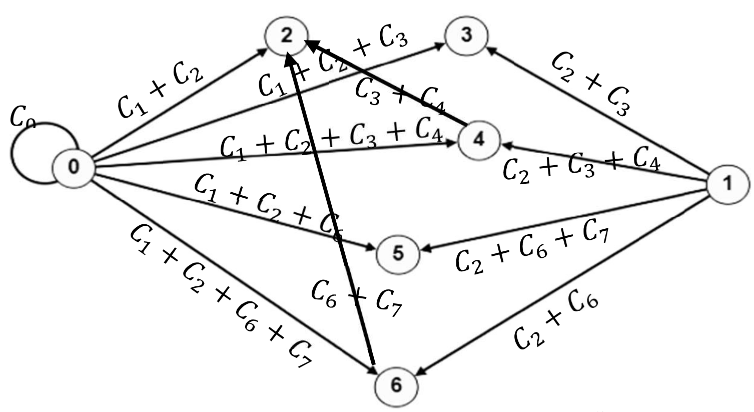

Figure 4. Reprsentation Graph of Matrix $\tilde{A}$

The next step is, the critical circuit of the graph will be determined using the weighted average of the maximum circuit. In the graph in **Figure 4**, it is very clear that there is only one path, namely 0 – 0 with a weight of C_0. This result shows us that the critical circuit of the system formed by the Timed Petri Net in **Figure 3** is only determined by one place, i.e. place P_0 with holding times C_0.

4. CONCLUSSION

From the findings above, it can be concluded that the entire air security process is very dependent on the readiness of the radar in monitoring the area. This is because the critical circuit obtained only depends on 1 place and 1 holding time. For future research, it is possible to deepen the flow process and workings of the air security system in more detail and take into account all possibilities so that more accurate results can be obtained.

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